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The contact magnetic flow in 3D Sasakian manifolds

J L Cabrerizo, M Fernández and J S Gómez

Department of Geometry and Topology, University of Seville, Ap 1160, Seville-41080, Spain

E-mail: jaraiz@us.es, mafernan@us.es and jsalvador@us.es

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Abstract
We first present a geometrical approach to magnetic fields in three-dimensional Riemannian manifolds, because this particular dimension allows one to easily tie vector fields and 2-forms. When the vector field is divergence free, it defines a magnetic field on the manifold whose Lorentz force equation presents a simple and useful form. In particular, for any three-dimensional Sasakian manifold the contact magnetic field is studied and the normal magnetic trajectories are determined. As an application, we consider the three-dimensional unit sphere, where we prove the existence of closed magnetic trajectories of the contact magnetic field, and that this magnetic flow is quantized in the set of rational numbers.

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1. Introduction

The motion of a physical system in a semi-Riemannian manifold, (M^n, g), is determined by the principle of the least action (also called the Maupertuis principle), and can be found by minimizing a certain Lagrangian (the action functional). Thus, the geodesic equations are second-order nonlinear differential equations and are commonly presented in the form of Euler–Lagrange equations of motion. By using the Hamilton–Jacobi approach to the geodesic equation, this statement can be given a very intuitive meaning: geodesics describe the motions of particles that are not experiencing any forces. Geodesics can also be defined as extremal curves for the following action energy functional:

$$E(\gamma) = \frac{1}{2}\int g(\gamma'(t), \gamma'(t))\,\mathrm{d}t,$$

where g is a Riemannian (or pseudo-Riemannian) metric. The defining property of geodesics is not that they minimize global lengths, but rather just local ones. The geodesic equation can then be obtained as the Euler–Lagrange equations of motion for this action.

In this paper, we study the normal trajectories of a charged particle in the presence of a time-independent magnetic field in 3D manifolds, so that in physical terminology our approach belongs to the classical magnetostatic theory [17]. In section 2, we first introduce some definitions and examples and we recall that the problem of characterizing magnetic flowlines in 2D manifolds (surfaces) from a global variational principle for a magnetic field F was solved in [3] for a certain class of magnetic fields, namely, the so-called Gaussian magnetic fields. The associated Lorentz force equation of these magnetic fields corresponds with the field equations of a variational problem that describe the massive relativistic bosons. On the other hand, for magnetic fields associated with a global potential $\omega \in \Lambda^1(M^n)$, i.e., $F = \mathrm{d}\omega$ on (M^n, g), we exhibit the global action that characterizes the magnetic trajectories of (M^n, g, F). In particular, this would happen if the second De Rham co-homology group vanishes, $\mathbf{H}^2(M^n) = 0$. But there are other interesting backgrounds where this class of magnetic fields arises. In fact, the magnetic field $F = \mathrm{d}\omega$ associated with a potential gauge ω on principal circle bundles $P(M^n, \mathbb{S}^1)$, the Kähler (uniform) magnetic field $F = k\Omega_J$, on Kähler manifolds (M^n, J, g) [1, 10], or the contact magnetic field F_ξ on Sasakian manifolds $(M^{2n+1}, \varphi, \xi, g)$ are some of them.

In section 3, 3D Riemannian manifolds (M^3, g) are considered, where the essential point is that the particular dimension enables one to study magnetic fields by using its associated vector field, which must be divergence free. Then, the Lorentz force equation takes an intuitive form. In section 4, we particularize to the case of Killing vector fields (which are divergence free), and a conservation law arises between their trajectories and the associated magnetic flowlines. In section 5, we consider Sasakian 3D manifolds, where the contact (or Reeb) vector field ξ is a global unit Killing vector field, and consequently it defines a magnetic field F_ξ. Then, we show that the Lorentz force ϕ of F_ξ and the tensor field φ of the Sasakian structure satisfy $\phi \equiv \varphi$. This fact makes it easy to prove that the normal magnetic trajectories of F_ξ are helices with axis ξ.

Finally, in section 6, the existence of periodic magnetic flowlines is investigated. This belongs to a class of non-trivial problems known classically as the *closed curve problem*. The solution is obtained for the 3D unit sphere $\mathbb{S}^3$, which is a Sasakian model space with the usual induced structure from $\mathbb{R}^4$.

2. Preliminaries and generalities

Let (M^n, g) be a nD Riemannian manifold. A closed 2-form F on M^n is said to be a *magnetic field*. In such a case, we shall write (M^n, g, F). Examples of magnetic fields are the following.

Example 2.1. *The curl magnetic field.* Let $V \in \mathfrak{X}(M^n)$ be an arbitrary vector field and ∇ the Levi-Civita covariant derivative of (M^n, g). The associated *curl* 2-form curl(V) is defined (see [13], p 95) by

$$\mathrm{curl}(V)(X, Y) = g(\nabla_X V, Y) - g(\nabla_Y V, X),$$

for all $X, Y \in \mathfrak{X}(M^n)$, which is an exact 2-form. In fact, if $V^\flat$ denotes the g-equivalent 1-form of V, then $\mathrm{curl}(V) = \mathrm{d}V^\flat$.

Example 2.2. *The magnetic field associated with a potential gauge.* Magnetic fields are also related to connections on principal circle bundles $P(M^n, \mathbb{S}^1)$. In fact, if ω denotes a potential gauge (i.e., a connection 1-form) on P, the strength or curvature 2-form is defined, via the structure equation, by $\mathrm{d}\omega = \mathbf{\Omega}$. Thus we get an associated magnetic field $F = \Omega$ on M^n.

The *Lorentz force* of a magnetic field F on (M^n, g) is defined to be the operator ϕ given by

$$g(\phi(X), Y) = F(X, Y),$$

for all $X, Y \in \mathfrak{X}(M)$. The *magnetic trajectories* of F are curves γ in M^n that satisfy the Lorentz equation

$$\nabla_{\gamma'}\gamma' = \phi(\gamma'). \tag{1}$$

Since the Lorentz force is skew symmetric we have

$$\frac{\mathrm{d}}{\mathrm{d}t} g(\gamma', \gamma') = 2g(\nabla_{\gamma'}\gamma', \gamma') = 0,$$

that is, magnetic curves have constant speed $v(t) = \|\gamma'(t)\| = v_0$. When the magnetic curve $\gamma(t)$ is arc-length parametrized ($v_0 = 1$), it is called a *normal magnetic curve*.

A special class of magnetic fields on a Riemannian manifold is that defined by the family of parallel 2-forms ($\nabla F = 0$), which are called *uniform magnetic fields*, and they play an important role in the classical Landau–Hall problem [3]. Note that they correspond with parallel Lorentz forces ($\nabla\phi = 0$).

The existence and uniqueness of geodesics remain true when one considers magnetic curves associated with an arbitrary magnetic field [3].

A very special case of a magnetic field appears when it is globally defined by an exact 2-form $F = \mathrm{d}\omega$. But, in general, this potential always exists at least locally. Denote by Γ_{pq} the space of smooth curves that connect two fixed points p, q of M^n. Now, we consider the action $\mathcal{L} : \Gamma_{pq} \to \mathbb{R}$ defined by

$$\mathcal{L}(\gamma) = \frac{1}{2}\int_\gamma g(\gamma', \gamma')\,\mathrm{d}t + \int_\gamma \omega(\gamma')\,\mathrm{d}t. \tag{2}$$

The tangent space of Γ_{pq} in γ is made up of the smooth vector fields, Z, along γ that vanish at p and q. To compute the extremals of (2), we first observe that

$$Z(\omega(\gamma')) = \gamma'(\omega(Z)) - g(\phi(\gamma'), Z).$$

Now, a standard computation involving integration by parts allows one to compute the first variation of this action to be

$$(\delta\mathcal{L})(\gamma)[V] = \int_\gamma g\left(\nabla_{\gamma'}\gamma' - \phi(\gamma'), V\right)\mathrm{d}t + [\omega(Z)]_{\partial\gamma}.$$

Since $[\omega(Z)]_{\partial\gamma} = 0$, we get that $(\delta\mathcal{L})(\gamma)[V] = 0$ for any $V \in T_\gamma\Gamma_{pq}$, if and only if γ is a solution of the Lorentz equation (1). Therefore, the Lorentz equation is indeed the Euler–Lagrange equation associated with the functional $\mathcal{L}$.

3. Magnetic fields in dimension 3

Dimension 3 is an interesting case for studying magnetic fields. In fact, there are several important facts which allow us to make a slightly different approach to their treatment.

Let (M^3, g) be an oriented 3D Riemannian manifold with volume 3-form Ω_3. First, 2-forms and vector fields may be considered as the same thing. In fact, if F is a 2-form F on (M^3, g), the Hodge star operator $\star$ provides a 1-form $\star F$ and hence the g-equivalent vector field $(\star F)^\sharp \in \mathfrak{X}(M^3)$. The converse trip starts with a vector field $V \in \mathfrak{X}(M^3)$, then consider its g-equivalent 1-form $V^\flat$ and then take its star, $\star V^\flat$. In this way, one obtains a 2-form which can also be written, using the interior contraction, as $\star V^\flat = i_V\Omega_3$. This defines a one-to-one map between 2-forms and vector fields.

On the other hand, note that magnetic fields mean divergence free vector fields. In fact, it is well known that the Lie derivative of the volume form Ω_3 satisfies $\mathcal{L}_V \Omega_3 = d(i_V \Omega_3) = \text{div}(V)\Omega_3$. This means that the 2-form $\star V^\flat = i_V \Omega_3$ is closed if and only if $\text{div}(V) = 0$, i.e., the volume element is invariant by the local flows of V. This allows us to regard magnetic fields in dimension 3 as divergence free vector fields. In particular, uniform magnetic fields correspond to parallel vector fields.

The 3D case gives us the possibility of finding out a suitable expression for the Lorentz force $\phi : T(M^3) \longrightarrow T(M^3)$, namely, this Lorentz force can be viewed in terms of the cross product as follows.

Definition 3.1. *The cross product of any two vector fields* $X, Y \in \mathfrak{X}(M^3)$ *is the vector field* $X \wedge Y \in \mathfrak{X}(M^3)$ *defined by*

$$g(X \wedge Y, Z) = \Omega_3(X, Y, Z). \tag{3}$$

Proposition 3.2. *Let* $V \in \mathfrak{X}(M^3)$ *with* $\text{div}(V) = 0$. *Then, the Lorentz force* ϕ *of the magnetic field* $F_V = i_V \Omega_3$ *is given by*

$$\phi(X) = V \wedge X. \tag{4}$$

Proof. It suffices to see that for any $Y \in \mathfrak{X}(M^3)$ equation (3) gives

$$g(V \wedge X, Y) = \Omega_3(V, X, Y) = (i_V \Omega_3)(X, Y) = F_V(X, Y) = g(\phi(X), Y).$$

□

Consequently, the Lorentz force equation (1) that provides the magnetic flow can be written as

$$\nabla_{\gamma'} \gamma' = V \wedge \gamma'. \tag{5}$$

In particular, this equation shows that any integral curve of the magnetic field V is a magnetic trajectory if and only if it is a geodesic.

It is clear that the cross product on a three-dimensional oriented Riemannian manifold satisfies the following two identities:

$$X \wedge (Y \wedge Z) = g(X, Z)Y - g(X, Y)Z,$$

$$g(X \wedge Y, X \wedge Z) = g(X, X)g(Y, Z) - g(X, Y)g(X, Z).$$

Remark 3.3.

(i) A geometrical construction similar to that shown in equation (3) has been used to define special almost contact structures on seven-dimensional manifolds endowed with a 2-fold vector cross product [12]. On the other hand, the cross product defined by equation (4) is an example of the *r*-fold cross product on manifolds introduced by Brown and Gray (see [6, 8]).

(ii) The Hall effect is a classical phenomenon for uniform magnetic fields in Euclidean space, and so in a free of gravity environment. Now, after equation (4), we note that the Hall effect also appears in a more general context. For example, it applies to any magnetic field in $\mathbb{R}^3$, not necessarily uniform. Moreover, it also works in any Riemannian 3D space, (M^3, g), even with non-trivial gravity. Therefore, we have that when an electric current flow, X, moves through a conductor (M^3, g) and is perpendicular to an applied magnetic field V, it experiences a force, the Lorentz force given by (4), acting normal to both directions and it moves in response to this force and the force is affected by its internal electric field.

4. Killing magnetic fields in 3D manifolds

Killing vector fields on a Riemannian manifold, (M^n, g), are those generating local flows of isometries, that is, $K \in \mathfrak{X}(M^n)$ is Killing if and only if $\mathcal{L}_K g = 0$, or equivalently, ∇K is a skew-symmetric operator, $g(\nabla_X K, Y) + g(X, \nabla_Y K) = 0$.

It is clear that any Killing vector field on (M^n, g) is divergence free. Consequently, if $n = 3$, then every Killing vector field K defines a magnetic field $F_K = i_K \Omega_3$ which will be called a *Killing magnetic field*. In particular, uniform magnetic fields, $\nabla K = 0$, are obviously Killing. Therefore, the class of Killing magnetic fields constitutes an important family of magnetic fields.

Besides the conservation law which asserts that the speed of any magnetic trajectory is constant, we now prove that the magnetic trajectories of Killing magnetic fields in dimension 3 have another additional conservation law.

Lemma 4.1. *A magnetic field F_K in a 3D Riemannian manifold is Killing if and only if for any magnetic curve γ of F_K the product $g(K, \gamma')$ is a constant along γ.*

Proof. In fact, if K is Killing and γ is a magnetic trajectory of F_K, then

$$\frac{\mathrm{d}}{\mathrm{d}t} g(K, \gamma') = g(\nabla_{\gamma'} K, \gamma') + g(K, \nabla_{\gamma'} \gamma') = 0.$$

Conversely, for $p \in M^3$ and $v \in T_p M^3$, let γ be a magnetic trajectory of F_K such that $\gamma(0) = p$, $\gamma'(0) = v$. We have

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} g(K, \gamma') = g(\nabla_{\gamma'} K, \gamma') + g(K, K \wedge \gamma') = g(\nabla_{\gamma'} K, \gamma').$$

Therefore, $g(\nabla_v K, v) = 0$, which means that K is Killing. □

The Killing vector fields K with constant length $\|K\| = g(K, K)^{1/2}$ are called *infinitesimal translations* and they are characterized by the following property: *a Killing vector field V is an infinitesimal translation if and only if their trajectories are geodesics.* A well-known obstruction to the existence of infinitesimal translations on a Riemannian manifold is obtained in terms of the Ricci curvature. In fact, for a Killing vector field of constant length one has $0 = \triangle \frac{1}{2} \|K\|^2 = \|\nabla K\|^2 - \mathrm{Ric}(K, K)$. Therefore, a Riemannian manifold with negative definite Ricci tensor does not admit a non-trivial infinitesimal translation. Finally, obviously a non-trivial infinitesimal translation has no zero. Therefore, if it exists on a compact manifold then its Euler number must be zero.

A very simple example is the Euclidean 3D space $(\mathbb{R}^3, g_{\mathbb{R}^3})$ and the magnetic field defined by the vector field $K = \partial_z$, which is an infinitesimal translation. It is well known that the trajectories of $(\mathbb{R}^3, g_{\mathbb{R}^3}, F_{\partial_z})$ are helices with axis ∂_z, that is, $\gamma(t) = (x_0 + a \cos t, y_0 + a \sin t, z_0 + bt)$, where $(x_0, y_0, z_0) \in \mathbb{R}^3$ and $a, b \in \mathbb{R}$.

5. The contact magnetic field

In order to study a more interesting background, namely, the 3D unit sphere $\mathbb{S}^3$, we now briefly recall the notion of an important class of 3D manifolds having a distinguished Killing magnetic field: the *Sasakian manifolds* [5], which are a particular type of *almost contact metric manifolds*. Sasakian manifolds constitute in some sense an odd-dimensional analog of Kähler manifolds.

For a given odd-dimensional manifold M^{2n+1} an *almost contact structure* on M^{2n+1} is triple (φ, ξ, η) where φ is a field of endomorphisms of the tangent spaces, a global vector field ξ and 1-form η such that

$$\eta(\xi) = 1, \qquad \varphi^2(X) = -X + \eta(X)\xi, \tag{6}$$

for any $X \in \mathfrak{X}(M^{2n+1})$. As a consequence of equations (6), we have also $\varphi(\xi) = 0$ and $\eta \cdot \varphi = 0$. Moreover, φ has rank $2n$.

A Riemannian metric g on the almost contact manifold $(M^{2n+1}, \varphi, \xi, \eta)$ is said to be *adapted or compatible* [5] if for all $X, Y \in \mathfrak{X}(M^{2n+1})$ the following equation is satisfied:

$$g(\varphi(X), \varphi(Y)) = g(X, Y) - \eta(X)\eta(Y). \tag{7}$$

Then $(M^{2n+1}, \varphi, \xi, \eta, g)$ is called an *almost contact metric manifold*. An immediate consequence is that $g(\xi, \xi) = 1$ and η is the covariant form of ξ, that is, $\eta(X) = g(\xi, X)$. The fundamental 2-form Φ of the structure is defined by $\Phi(X, Y) = g(X, \varphi(Y))$. When the fundamental 2-form satisfies $\Phi = \mathrm{d}\eta$, then $(M^{2n+1}, \varphi, \xi, \eta, g)$ is called a *contact metric manifold*.

In a contact metric manifold, the integral curves of ξ are geodesics. A contact metric manifold such that the vector field ξ is a Killing vector field with respect to g is called a *K-contact manifold*. The first basic property of a *K*-contact manifold is that

$$\nabla_X \xi = -\varphi(X). \tag{8}$$

A *Sasakian manifold* is a contact metric manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ such that

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X, \tag{9}$$

for all $X, Y \in \mathfrak{X}(M^{2n+1})$. It is easy to see that any Sasakian manifold is *K*-contact. In dimension 3, the converse is true.

Now we shall prove that in any 3D almost contact metric manifold (in particular, Sasakian manifolds) the tensor field φ is completely determined.

Proposition 5.1. *Let (M^3, φ, ξ, g) be a 3D almost contact metric manifold. Then, the structure (1,1)-tensor field φ is given by*

$$\varphi(X) = \xi \wedge X, \tag{10}$$

for any $X \in \mathfrak{X}(M^3)$.

Proof. Let G be a local coordinate neighborhood in M^3 and take a unit vector field U on G orthogonal to ξ. Then $\varphi(U)$ is a unit vector field which is orthogonal to U and ξ and hence $\{\xi, U, \varphi(U)\}$ is a local orthonormal frame which is called a φ-basis [5]. Define the orientation on M^3 such that the volume element Ω_3 satisfies $\Omega_3(\xi, U, \varphi(U)) = 1$. First, it is clear that $0 = \varphi(\xi) = \xi \wedge \xi$. On the other hand, as $\varphi(U)$ is a unit vector field orthogonal to ξ and U, then $\varphi(U) = \pm\xi \wedge U$. But $g(\xi \wedge U, \varphi(U)) = \Omega_3(\xi, U, \varphi(U)) = 1$, and hence $\varphi(U) = \xi \wedge U$. Finally, we have $\varphi(\varphi(U)) = -U = -\varphi(U) \wedge \xi = \xi \wedge \varphi(U)$. Therefore, $\varphi(X) = \xi \wedge X$ for any $X \in \mathfrak{X}(M^3)$. □

Recall that in a Sasakian manifold (M^3, φ, ξ, g) the contact (or Reeb) vector field ξ is Killing. Therefore, we have naturally defined on M^3 the Killing magnetic field $F_\xi = i_\xi \Omega_3$, which will be called the *contact magnetic field* of the Sasakian 3D manifold. The Lorentz force ϕ of this magnetic field and the tensor φ of the Sasakian structure are tied together as follows.

Corollary 5.2. *Let (M^3, φ, ξ, g) be a 3D Sasakian manifold. Then, the structure tensor φ and the Lorentz force ϕ of the Killing magnetic field F_ξ satisfy*

$$\varphi \equiv \phi. \tag{11}$$

Proof. It follows immediately from equation (3) and proposition 5.1. □

Next we shall determine the normal magnetic trajectories of the contact magnetic field F_ξ.

Theorem 5.3. *Let (M^3, φ, ξ, g) be a Sasakian manifold. The normal flowlines $\gamma(t)$ of the contact magnetic field F_ξ are the helices of axis ξ with constant curvature $\kappa_0 = \sin\theta_0 > 0$ and constant torsion $\tau_0 = 1 + \cos\theta_0$, where θ_0 is the (constant) angle between $\gamma'(t)$ and $\xi_{\gamma(t)}$.*

Proof. Suppose $\gamma(t)$ is a normal magnetic curve of F_ξ. Since ξ is a unit vector field, lemma 4.1 says that $g(\xi_{\gamma(t)}, \gamma'(t)) = \cos\theta_0$, $0 < \theta_0 < \pi$.

On the other hand, by corollary 5.2 we have $\phi(\gamma') \equiv \varphi(\gamma') = \xi \wedge \gamma'$. Consequently, the Lorentz equation reads

$$\nabla_{\gamma'}\gamma' = \xi \wedge \gamma'. \tag{12}$$

Let $\{T(t) = \gamma'(t), N(t), B(t)\}$, $\kappa(t)$, $\tau(t)$ be the Frenet frame, the (geodesic) curvature and the torsion of $\gamma(t)$, respectively. The first Frenet equation for γ is given by

$$\nabla_{\gamma'}\gamma' = \kappa N. \tag{13}$$

Then $\kappa N = \xi \wedge \gamma'$, and hence

$$\kappa^2 = g(\xi \wedge \gamma', \xi \wedge \gamma') = 1 - \cos^2\theta_0 = \sin^2\theta_0. \tag{14}$$

Thus $\kappa(t) = \kappa_0 = \sin\theta_0 > 0$ is a constant. The binormal vector of γ is defined by

$$B = \gamma' \wedge N = \frac{1}{\kappa_0}\gamma' \wedge (\xi \wedge \gamma') = \frac{1}{\kappa_0}(\xi - \cos\theta_0\gamma'). \tag{15}$$

Now, equation (15) combined with the third Frenet equation $\nabla_{\gamma'}B = -\tau N$ yields

$$\frac{1}{\kappa_0}(\nabla_{\gamma'}\xi - \cos\theta_0\nabla_{\gamma'}\gamma') = -\tau\left(\frac{1}{\kappa_0}\xi \wedge \gamma'\right), \tag{16}$$

and then equation (8) gives

$$-\varphi(\gamma') - \cos\theta_0\varphi(\gamma') = -\tau\xi \wedge \gamma' = -\tau\varphi(\gamma'),$$

which yields $\tau = \tau_0 = 1 + \cos\theta_0$. Therefore, γ is a helix (curvature and torsion are constant) with axis ξ.

Conversely, assume that $\gamma(t)$ is an arc-length parametrized helix with axis ξ, constant curvature $\kappa_0 = \sin\theta_0 > 0$ and constant torsion $\tau_0 = 1 + \cos\theta_0$, $0 < \theta_0 < \pi$, where θ_0 is the angle between $\gamma'(t)$ and $\xi_{\gamma(t)}$. Then, the covariant derivative of $g(\gamma', \xi) = \cos\theta_0$ along γ gives

$$0 = g(\nabla_{\gamma'}\gamma', \xi) + g(\gamma', \nabla_{\gamma'}\xi) = g(\kappa_0 N, \xi) + g(\gamma', -\varphi(\gamma')) = \kappa_0 g(N, \xi),$$

where we have used again the fundamental equation on Sasakian manifolds $\nabla_X\xi = -\varphi(X)$. Thus, N is orthogonal to ξ and therefore $N = \lambda\xi \wedge \gamma'$, where $\lambda(t)$ is a non-vanishing function. Computing modules on both sides of this equation we obtain $1 = |\lambda(t)|\sin\theta_0$ and hence we conclude that $\lambda(t) = \lambda_0 \neq 0$ is a constant. Thus, we have that

$$B = \gamma' \wedge N = \lambda_0\gamma' \wedge (\xi \wedge \gamma') = \lambda_0(\xi - \cos\theta_0\gamma').$$

A substitution of this formula for B in the third Frenet equation $\nabla_{\gamma'} B = -\tau_0 N$ yields

$$\lambda_0(\nabla_{\gamma'}\xi - \cos\theta_0 \nabla_{\gamma'}\gamma') = -\tau_0\lambda_0\xi \wedge \gamma' = -\tau_0\lambda_0\varphi(\gamma').$$

But since $\nabla_{\gamma'}\xi = -\varphi(\gamma')$, the last equation then reads

$$-\varphi(\gamma') - \cos\theta_0 \nabla_{\gamma'}\gamma' = -\tau_0\varphi(\gamma'),$$

or equivalently,

$$\nabla_{\gamma'}\gamma' = \frac{\tau_0 - 1}{\cos\theta_0}\varphi(\gamma') = \varphi(\gamma') = \phi(\gamma').$$

Therefore $\nabla_{\gamma'}\gamma' = \phi(\gamma')$, and this proves that γ is a normal flowline of the contact magnetic field F_ξ. □

Remark 5.4.

(a) Since every 3D K-contact manifold is Sasakian, theorem 5.3 is also true for a 3D K-contact manifold.

(b) As we noticed, the limit cases $\theta_0 = 0, \pi$ mean that γ is an integral curve of ξ. But the trajectories of ξ are then geodesics ($\nabla_\xi \xi = 0$), which fits with our formula $\kappa = \sin\theta_0 = 0$ for the geodesic curvature in theorem 5.3.

(c) In 1971, Martinet proved that every compact orientable 3D manifold carries a contact structure [11]. On the other hand, the topology of three-dimensional Sasakian manifolds is well known in the compact case. In fact, any compact Sasakian manifold is a Seifert fibration but the Sasakian structures can be explicitly described [4].

6. Periodic orbits of the contact magnetic field in the 3D sphere

It is a well-known conjecture of Weinstein [19] that on a *compact* contact manifold M^{2n+1} satisfying $H^1(M^{2n+1}, R) = 0$, the vector field ξ must have a closed orbit. In a recent paper, Taubes [16] proved that the conjecture is true but the second hypothesis is superfluous, that is, on any compact oriented 3D contact manifold the vector field ξ has a closed orbit.

Associated with this interesting problem, our aim now is to investigate the existence of periodic magnetic flowlines of the contact magnetic field F_ξ. This belongs to a class of non-trivial problems known classically as the *closed curve problem*. The solution is given in terms of data which are encoded in the underlying geometry governing the model.

Let $\mathbb{S}^3 = \{z = (z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$ be the unit sphere endowed with its usual metric $g_{\mathbb{S}^3}$ induced from $\mathbb{R}^4 \equiv \mathbb{C}^2$. Now, this induced differentiable structure on $\mathbb{S}^3$ is Sasakian. In fact, consider the vector field, ξ on $\mathbb{S}^3$, defined by $\xi_z = iz = (iz_1, iz_2) \in T_z\mathbb{S}^3$ for any $z \in \mathbb{S}^3$. Certainly, this is a Killing vector field with constant length 1 and so, an infinitesimal translation on $\mathbb{S}^3$. In fact, ξ generates a global $\mathbb{S}^1$-action defined by $\{\psi_t : \mathbb{S}^3 \to \mathbb{S}^3 : t \in \mathbb{R}\}$ with $\psi_t(z) = \mathrm{e}^{\mathrm{i}t}z$, which are isometries of $\mathbb{S}^3$. Thus, the Hopf map: $\pi : \mathbb{S}^3 \to \mathbb{S}^2$ defines a principal circle (or $U(1)$) bundle.

On the other hand, we define a (1,1)-tensor field φ and a 1-form η on $\mathbb{S}^3$ such that for any $X \in T_z\mathbb{S}^3$ the vector field iX splits into tangential and normal components as

$$iX = \varphi(X) - \eta(X)z. \tag{17}$$

Then applying i to each side of equation (17) we have

$$-X = \varphi^2(X) - \eta(\varphi(X))z - \eta(X)\xi$$

and hence

$$\varphi^2(X) = -X + \eta(X)\xi, \qquad \eta(\varphi) = 0.$$

Taking $X = \xi$ in equation (17) we have $\varphi(\xi) = 0$ and $\eta(\xi) = 1$. Thus, (φ, ξ, η) is the usual contact structure of $\mathbb{S}^3$, induced from $\mathbb{C}^2$, which is also Sasakian.

Consequently, we use theorem 5.3 to obtain that the normal trajectories of the contact magnetic field F_ξ in the 3D sphere are helices with axis ξ. It is well known that helices in Euclidean space can be regarded as geodesics in right cylinders (of course the converse also holds). Now we show a similar result for helices in the 3D sphere.

If we take the standard 2-sphere $\mathbb{S}^2(1/2)$ of radius 1/2, then the Hopf map $\pi : \mathbb{S}^3(1) \to \mathbb{S}^2(1/2)$ becomes also a Riemannian submersion (see [13], p 212). First, for any given curve $\beta(u)$ in $\mathbb{S}^2(1/2)$, its complete lift via π to $\mathbb{S}^3(1)$ is a surface $\mathbf{H}_\beta = \pi^{-1}(\beta)$, which is proved to be a flat surface [14] and is called the *Hopf tube* over β. Moreover, an immersed surface N in $\mathbb{S}^3$ is $\mathbb{S}^1$-invariant if and only if $N = \mathbf{H}_\gamma = \pi^{-1}(\gamma)$ for some immersed curve γ in $\mathbb{S}^2$ (i.e., N is a Hopf tube).

When β is a closed curve in $\mathbb{S}^2(1/2)$, the Hopf tube $\mathbf{H}_\beta$ is a *Hopf torus*. In particular, for a geodesic circle $\beta(u)$ with curvature ρ, the Hopf torus $\mathbf{H}_\beta = \pi^{-1}(\beta)$ is a flat torus with constant mean curvature, $H = \rho/2$ (the Clifford torus is obtained when β is a great circle). All these tori can be naturally parametrized from the following Riemannian covering map:

$$\Phi : \mathbb{R}^2 \to \mathbf{H}_\beta \subset \mathbb{S}^3, \qquad \Phi(u, t) = \mathrm{e}^{\mathrm{i}t}\bar{\beta}(u),$$

where $\bar{\beta}$ stands for a horizontal lift of β. Therefore, the coordinate families of curves are the fibers (u = constant) and the horizontal lifts of β (t = constant). On the other hand, each geodesic γ of $\mathbf{H}_\beta$ is determined from its slope $\sigma = b/a$ measured with respect to Φ, that is,

$$\gamma(s) = \Phi(as, bs), \qquad \gamma'(s) = a\Phi_u + b\Phi_t.$$

It is easy to see that γ is a helix of $\mathbb{S}^3$ with curvature and torsion given, respectively, by

$$\kappa = \frac{\rho + 2\sigma}{1 + \sigma^2}, \qquad \tau = \frac{1 - \sigma\rho - \sigma^2}{1 + \sigma^2},$$

and making a constant angle with ξ, i.e., it is a helix in $\mathbb{S}^3$ with axis ξ.

Moreover, the converse of the above fact also holds. Given any helix γ of $\mathbb{S}^3$ with axis ξ, curvature $\kappa \neq 0$ and torsion τ, we consider the geodesic circle β of $\mathbb{S}^2(1/2)$ with curvature

$$\rho = \frac{\kappa^2 + \tau^2 - 1}{\kappa},$$

and then, if we pick in its Hopf torus, $\mathbf{H}_\beta$, the geodesic determined by the slope

$$\sigma = \frac{1 - \tau}{\kappa},$$

one obtains a curve with curvature κ and torsion τ which obviously is congruent to γ in $\mathbb{S}^3$.

As a consequence, we have proved the following.

Corollary 6.1. *The magnetic trajectories of* $(\mathbb{S}^3, g_{\mathbb{S}^3}, F_\xi)$ *are the geodesics of the Hopf tori in* $\mathbb{S}^3$ *constructed over geodesics circles in* $\mathbb{S}^2(1/2)$.

Theorem 6.2. *The moduli space, up to similarities, of periodic magnetic curves of the contact magnetic field in the 3D sphere can be identified and so quantized in the set of rational numbers.*

Proof. It is not difficult to see that a geodesic γ is periodic if and only if its slope and the radius $r \in (0, 1/2)$ satisfy the following quantization principle:

$$r\sigma + \frac{1}{2}\sqrt{1 - 4r^2} = q \quad \text{is a rational number.} \tag{18}$$

Now, we can compute the moduli space of periodic orbits of the contact magnetic field defined by ξ. For this end, choose any point $z \in \mathbb{S}^3$ and a Hopf torus $\mathbf{H}_\beta$ with $z \in \mathbf{H}_\beta$ where β is a circle in $\mathbb{S}^2(1/2)$. Note that this circle and so its Hopf torus are unique up to similarities. Now, for any unit vector $v \in T_z(\mathbf{H}_\beta)$ with slope σ satisfying (18), we take

(i) the unique unit speed geodesic γ of $\mathbf{H}_\beta$ through z in the direction of v, which we know is closed;

(ii) the unique unit speed normalized magnetic curve δ of $(\mathbb{S}^3, g_{\mathbb{S}^3}, F_\xi)$ through z in the direction of v.

Since every magnetic curve of $(\mathbb{S}^3, g_{\mathbb{S}^3}, V)$ is a helix and so a geodesic of a Hopf torus, both curves coincide. □

Remark 6.3. The Hopf fibration was introduced around 1931 as a purely mathematical idea, but however, it also occurs in at least seven different situations in theoretical physics (two-level quantum systems, harmonic oscillator, Taub-NUT space, twistors–Robinson congruences, helicity representations, magnetic monopoles and Dirac equation) [18].

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